Covariant Impulse Approximation for the study of the internal structure of composite particles

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Abstract

We present a brief review on the Impulse Approximation method to study processes of scattering off composite particles. We first construct the model in a non-relativistic fashion that enables us to extend the model to a covariant Impulse Approximation, which is needed for the study of high momentum transfer processes.

1 Introduction

The possibility of investigating the internal structure of composite bound systems by means of the scattering processes is based on the approximated validity of the following hypothesis, denoted as Impulse Approximation (IA). The probing particle that makes a scattering off the bound state, only interacts with a single constituent at time. Moreover, the state of motion of the other particles remains unaltered, that is their momenta do not change, during the scattering process. For this reason, these particles are called "the spectators". This scheme can really work if the interaction of the probing particle with the constituents of the bound system is "weak", allowing for the perturbative treatment of the scattering process, how it is assumed in the previous description.
In practice, the IA has been successfully applied to the study of the electromagnetic and weak scattering processes. In particular, very relevant physical results have been obtained studying the elastic and inelastic form factors of the atomic nuclei, where a non-relativistic treatment is sufficient to give a consistent description of the scattering process.
On the other hand, for the study of lepton scattering off hadrons, where the non-relativistic model fails to reproduce the experimental results, specially at high

momentum transfer, a generalization of the Impulse Approximation is required. Aim of the present work is to introduce, in a didactic way, a model for the covariant generalization of the IA. More technical details are discussed elsewhere [1]. For simplicity and clarity we shall explicitly treat two and three hadron composite systems, for which accurate wave-functions can be constructed without great difficulties. However, our model can be straightforwardly generalized to the case of A constituents.

2 The non-relativistic model

The general ideas of the previously introduced impulse approximation have been used to define a non-relativistic model, that we shall discuss in the present section with the aim of fixing the starting point for its relativistic generalization.
Schematically, the Non Relativistic Impulse Aroximation NRIA, can be summarized by the following equation:

$$\bar{A}_{FI} = \langle \Psi_F | \sum_{i=1}^{A} \widehat{O}_i e^{i\vec{q}\vec{x}_i} | \Psi_I \rangle. \tag{2.1}$$

The quantity $\bar{A}_{FI}$, that is the Transition Amplitude, or better, the factor related to the bound state transition (being the leptonic part a well known quantity), is defined as the matrix-element, between the initial and final wave-functions of the bound system $|\Psi_{F/I}\rangle$, of the transition operator given by the sum of the transition operators of the single particle (denoted by the letter i). The exponential operator increases by the amount $\vec{q}$, the 3-momentum of the i-th struck constituent, while the other constituents, the spectators, do not change their momenta as required by the hypothesis of the IA. This property takes the form

$$e^{i\vec{q}\vec{x}_i} |\vec{p}_1, ..., \vec{p}_i, ..., \vec{p}_A\rangle = |\vec{p}_1, ..., \vec{p}_i + \vec{q}, ..., \vec{p}_A\rangle. \tag{2.2}$$

The single particle "current" operator has been synthetically denoted as $\widehat{O}_i$. Also, in equation (2.1) we have neglected, for simplicity, all tensor indices; for the same reason, isospin and/or charge operators have not been explicitly indicated.
In the non-relativistic quantum mechanics the wave function can be easily separated into a first factor that describes the internal motion and a plane-wave function of momentum $\vec{P}_{I/F}$ for the motion of the composite particle as a whole. One has

$$|\Psi_{F/I}\rangle = |\psi_{F/I}\rangle |\vec{P}_{F/I}\rangle. \tag{2.3}$$

Furthermore, the single particle operators $\vec{x}_i$ can be conveniently expressed in terms of the center of mass $\vec{R}$ and intrinsic coordinates. As anticipated, we shall treat explicitly the two and three body cases for constituents of equal mass, but the procedure can be generalized to A constituents without difficulties.

The definition of the intrinsic coordinates is not unique. In the following relations we report the most widely used choice, that allows for highlighting the permutational symmetries of the wave functions. For two particles we have:

$$\begin{aligned}
\vec{r} &= \vec{r}_1 - \vec{r}_2, \\
\vec{R} &= \frac{1}{2}(\vec{r}_1 + \vec{r}_2), \\
\vec{p} &= \frac{1}{2}(\vec{p}_1 - \vec{p}_2), \\
\vec{P} &= \vec{p}_1 + \vec{p}_2.
\end{aligned} \tag{2.4}$$

And for three particles:

$$\begin{aligned}
\vec{\rho} &= \frac{1}{\sqrt{2}}(\vec{r}_1 - \vec{r}_2), \\
\vec{\lambda} &= \frac{1}{\sqrt{6}}(\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3), \\
\vec{R} &= \frac{1}{3}(\vec{r}_1 + \vec{r}_2 + \vec{r}_3), \\
\vec{p}_\rho &= \frac{1}{\sqrt{2}}(\vec{p}_1 - \vec{p}_2), \\
\vec{p}_\lambda &= \frac{1}{\sqrt{6}}(\vec{p}_1 + \vec{p}_2) - \sqrt{\frac{2}{3}}\vec{p}_3, \\
\vec{P} &= \vec{p}_1 + \vec{p}_2 + \vec{p}_3.
\end{aligned} \tag{2.5}$$

Furthermore, in the following we shall consider composite systems of identical particles. For symmetry reasons the total matrix-element $\bar{A}_{FI}$ may be written as A-times the contribution of a single interacting constituent. For this we choose the constituent number 2 for the two-body case ($A = 2$) and the constituent number 3 for the three-body case ($A = 3$). Using the intrinsic operators of equations (2.4), (2.5), one obtain the conservation of the total momentum as shown in the following equation. For bound states of $k = 2, 3$ constituents we have

$$\bar{A}_{FI} = A_{FI}^{(k)} \delta^3(\vec{P}_F - \vec{P}_I - \vec{q}). \tag{2.6a}$$

Also, the intrinsic matrix-element or vertex factor has the form

$$A_{FI}^{(2)} = 2\langle \psi_F^{(2)} | \widehat{O}_2 e^{-i\vec{q}\frac{\vec{r}}{2}} | \psi_I^{(2)} \rangle, \tag{2.7a}$$

$$A_{FI}^{(3)} = 3\langle \psi_F^{(3)} | \widehat{O}_3 e^{-i\vec{q}\lambda\sqrt{\frac{2}{3}}} | \psi_I^{(3)} \rangle, \tag{2.7b}$$

for the two-and three-body cases, respectively.

We explicitly consider the electric charge density transition amplitudes that are obtained replacing in equations (2.7) the generic operator $\widehat{O}_i$ with the electric charge, that is

$$\widehat{O}_i = \widehat{e}_i,$$

with $i = 2, 3$ for equation (2.7a) and equation (2.7b), respectively.
For the intrinsic part of the electric current density transition amplitudes, one has

$$\widehat{O}_2 e^{-i\vec{q}\frac{\vec{r}}{2}} \longrightarrow -\frac{\hat{e}_2}{2m}\{\vec{p}, e^{-i\vec{q}\frac{\vec{r}}{2}}\}$$

for equation (2.7a), and

$$\widehat{O}_3 e^{-i\vec{q}\vec{\lambda}\sqrt{\frac{2}{3}}} \longrightarrow -\frac{\hat{e}_3}{2m}\sqrt{2}3\{\vec{p}_\lambda, e^{-i\vec{q}\vec{\lambda}\sqrt{\frac{2}{3}}}\},$$

for equation (2.7b). In the previous equations the constituent particle mass m has been introduced. A complete list of these electromagnetic current operators, including the relativistic corrections is given in ref. [2].
For the subsequent relativistic study, it is convenient to express the typical vertex factors of equations (2.4 b,c) in terms of the intrinsic momentum space wavefunctions:

$$A_{FI}^{(2)} = 2\int d^3p \psi_F^\dagger(\vec{p} - \frac{\vec{q}}{2})\hat{O}_2\psi_I(\vec{p}), \tag{2.8a}$$

$$A_{FI}^{(3)} = 3\int d^3p_\rho d^3p_\lambda \psi_F^\dagger \left(\vec{p}_\rho, p_\lambda - \sqrt{2/3}\vec{q}\right)\hat{O}_3\psi_I(p_\rho, p_\lambda). \tag{2.8b}$$

Concluding this section, we point out that the model that we have described is intrinsically non-relativistic. In fact, from a formal point of view, the starting point of equation (2.1) cannot be put in a covariant form being absent in the exponential operator any reference to the zero (time) components of the position vectors $\vec{x}_i$ and of the momentum transfer $\vec{q}$. Also, the separation of the intrinsic and CM variables of equations (2.4), (2.5) is basically non-relativistic. For the electromagnetic interactions, it is possible to include in the previous model all the relativistic corrections up to order c^{-2} (or $\frac{p^2}{m^2}$), by considering the Foldy Wouthuysen expansion of the electromagnetic current matrix-elements and adding the Lorentz boost corrections to the position and spin operators. This procedure, that belongs to the theoretical framework of the instant form relativistic quantum mechanics [3], has been applied to the study of the electromagnetic interactions of the hadrons, obtaining numerically relevant results for some low-momentum transfer observable, as the mean squared charge radius of the proton in the constituent quark model [4]. However, it has not been found a viable method to generalize such procedure for including the relativistic effects to all orders beyond the first order corrections.

3 The relativistic model

The NRIA described in the previous section, does not make use of the ideas of relativity. In the following we show that it is possible to define a relativistic three dimensional model, i.e. without introducing the time components of the dynamical

quantities, conserving the main requirements of the Impulse Approximation. For didactic reasons, we discuss the model in the case of scalar constituents. However, the generalization to the case of fermionic particles is straightforward, For the sake of clarity we point out that we observe and describe the scattering process in an "observation reference frame" -ORF- that may be for example the laboratory, the center of mass, the Breit, etc. On the other hand we assume that the bound state can be represented by a momentum space wave function, originally defined in its rest frame, that will be denoted by an asterisk (*). Note that whatever ORF we choose, the bound state will be in motion in the initial and/or in the final state of the scattering process, so that in the ORF, two different wave functions are observed, one for the initial state and another for the final state. As in the non-relativistic model, we make the hypothesis that only one particle at time interacts with the external field.

For identical particle systems, as before, we take the constituent 2 for the two-body case and the constituent 3 for the three body case. The total result is then obtained multiplying by A=2 and A=3, respectively. In the following we focus our attention on the vertex factor A_{FI}. For the calculation of this quantity we assume that the momenta of the spectators (i.e. the non interacting constituents) remain unchanged during the scattering process. As a specific assumption of the relativistic model, we put the momenta of the spectators "on the mass shell", that is,

$$p_i = (E(\vec{p}_i), \vec{p}_i), \qquad \text{with} \qquad E(\vec{p}_i) = \sqrt{\vec{p}_i^{\,2} + m^2}, \tag{3.1}$$

or equivalently, in any reference frame

$$p_i^2 = m^2, \tag{3.2}$$

with $i = 1$ and $i = 1, 2$ for the two-body and the three-body case respectively. Note that $A - 1$ momenta are required to describe the internal function of the bound system. Consequently, we use these momenta as the spatial integration variables of the ORF for the calculation of the vertex factor (i.e. matrix element) A_{FI}.We perform three dimensional covariant integrations

$$\int \frac{d^3 p_1}{E(\vec{p}_1)}, \qquad \int \frac{d^3 p_1}{E(\vec{p}_1)} \int \frac{d^3 p_2}{E(\vec{p}_2)}, \tag{3.3}$$

that ensure the covariance of the model. It means that, when the ORF is changed, also the observable quantities A_{FI} are changed according to the Lorentz tensor properties of the operators $\hat{O}_i$.

Given that the wave functions are originally defined in the bound system rest reference frame we use standard Lorentz transformations to determine the initial and final reference frame momenta

$$p^{*\alpha}_{iI/F} = \Lambda^{\alpha}_{\mu}(\vec{\beta}_{I/F}) p^{\mu}_i, \tag{3.4}$$

where $\alpha = 1, 2, 3$ denotes the spatial components of the rest reference frame momenta and as before $i = 1$ and $i = 1, 2$, for two- and three-body case. The Lorentz

transformations depend on the velocity of the initial/final state with respect to the ORF, that is

$$\vec{\beta}_{I/F} = \frac{\vec{P}_{I/F}}{\sqrt{M^2_{I/F} + \vec{P}^2_{I/F}}}. \tag{3.5}$$

Explicitly equation (3.4) takes the form

$$\vec{p}^*_i = \vec{p}_i + \frac{\vec{P}}{M}\left[\frac{\vec{P}\cdot\vec{p}_i}{E+M} - E_i\right] \tag{3.6}$$

Assuming that in the rest reference frame the sum of the spatial momenta of the constituents is zero,

$$\begin{aligned} \vec{p}^*_1 + \vec{p}^*_2 &= 0, \\ \vec{p}^*_1 + \vec{p}^*_2 + \vec{p}^*_3 &= 0, \end{aligned} \tag{3.7}$$

we can express the Jacobi $A-1$ momenta $\vec{p}$ and $\vec{p}_\rho$, $\vec{p}_\lambda$ as linear combinations of the $\vec{p}^*_{iI/F}$ determined by the Lorentz transformations of equation (3.6), with, as usual, $i=1$ and $i=1,2$ for the two- and three-body case, respectively.
We have almost all the ingredients to write down the vertex factors of the covariant model. We shall discuss in the following the normalization factors that are introduced. We have

$$A^{(2)}_{FI} = \frac{2}{J^{(2)}}\int \frac{d^3p_1}{E(\vec{p}_1)}\psi^\dagger(\vec{p}^*_{1F})\sqrt{E(\vec{p}^*_{1F})}\hat{O}_2\sqrt{E(\vec{p}^*_{1I})}\psi(\vec{p}^*_{1I}), \tag{3.8}$$

$$A^{(3)}_{FI} = \frac{3}{J^{(3)}}\int \frac{d^3p_1}{E(\vec{p}_1)}\frac{d^3p_2}{E(\vec{p}_2)}\psi^\dagger(\vec{p}^*_{1F},\vec{p}^*_{2F},)\sqrt{E(\vec{p}^*_{1F}E(\vec{p}^*_{2F})}\hat{O}_3\sqrt{E(\vec{p}^*_{1I})E(\vec{p}^*_{2I})}\psi(\vec{p}^*_{1I},\vec{p}^*_{2I}). \tag{3.9}$$

The use of intrinsic variables $\vec{p}^*_{iI/F}$ instead of the Jacobi momenta, makes necessary to introduce the normalization jacobians

$$J^{(2)} = 1 \tag{3.10a}$$

$$J^{(3)} = 3^{-3/2} \tag{3.10b}$$

Furthermore, due to the use of the covariant integrations, it has been necessary to introduce the invariant normalization factors

$$\sqrt{E(\vec{p}^*_{1I/F})}, \qquad \sqrt{E(\vec{p}^*_{1I/F})E(\vec{p}^*_{2I/F})},$$

to ensure that in the static limit, *i.e.* $\vec{P}_I = \vec{P}_F = 0$, $M_F = M_I$, the vertex factors reduce to the static mean value of the corresponding operators $\hat{O}_2$ and $\hat{O}_3$.
This point may be examined in a slightly different way, shading some light on the relativistic properties of the model. First, we observe that the boost of the wave

function is realized in the model by expressing the intrinsic momenta $\vec{p}_i^*$ in terms of the ORF momenta $\vec{p}_1$ and $\vec{p}_1$, $\vec{p}_2$ by means of the standard Lorentz transformation of equation (3.6).
We can write

$$\widetilde{\psi}(\vec{p}_i) = \psi(\vec{p}_i^*(\vec{p}_i)). \tag{3.11}$$

However the boosted wave function $\widetilde{\psi}(\vec{p}_i)$ of the previous equation is not normalized in a standard way, if one integrates over the $\vec{p}_i$. Straightforward use of the covariant integration rules show that

$$d^3p_i^* = d^3p_i \frac{E(\vec{p}_i^*)}{E(\vec{p}_i)}. \tag{3.12}$$

In consequence, the correctly normalized boosted wave functions are

$$\psi(\vec{p}_1) = \left[\frac{E(\vec{p}_1^*)}{E(\vec{p}_1)}\right]^{1/2} \psi(\vec{p}_1^*(\vec{p}_1)), \tag{3.13}$$

$$\psi(\vec{p}_1, \vec{p}_2) = \left[\frac{E(\vec{p}_1^*)E(\vec{p}_2^*)}{E(\vec{p}_1)E(\vec{p}_2)}\right]^{1/2} \psi(\vec{p}_1^*(\vec{p}_1), \vec{p}_2^*(\vec{p}_2)), \tag{3.14}$$

to be integrated with respect to d^3p_1 and $d^3p_1d^3p_2$, respectively. When a vertex factor is calculated, integrating with respect to the same ORF momenta, the product of the denominators of the previous equations give the factors $E(\vec{p}_1)$ and $E(\vec{p}_1)E(\vec{p}_2)$, that are the denominators of the covariant integration, while the numerators give the invariant normalization factors thar have been previously introduced.
This procedure of boosting the wave function, *i.e.* by transforming the on mass shell momenta, with the correct normalization factors, represents a realization of the Point Form Relativistic Quantum Mechanics [1, 3].
In the model, the effects of the interaction that binds the system, are only parametrically contained in the velocity parameter $\vec{\beta}_{I/F}$, while the momenta of the particles are on mass shell.
As a check of the consistency of the model we calculate the non-relativistic limit of the vertex factor, showing that the same result as the NRIA is obtained. First, we note that the factors

$$\frac{\sqrt{E(\vec{p}_{iF}^*)E(\vec{p}_{iI}^*)}}{E(\vec{p}_i)}$$

with $i = 1$ and $i = 1, 2$, reduce to unity in the NR limit. The vertex factors of equations (3.8) y (3.9) take the form

$$A_{FI}^{(2)NR} = \frac{2}{J^{(2)}} \int d^3p_1 \psi^\dagger(\vec{p}_{1F}^{*NR}) \hat{O}_2 \psi(\vec{p}_{1I}^{*NR}), \tag{3.15}$$

$$A_{FI}^{(3)NR} = \frac{3}{J^{(3)}} \int d^3p_1 d^3p_2 \psi^\dagger(\vec{p}_{1F}^{*NR}, \vec{p}_{2F}^{*NR}) \hat{O}_2 \psi(\vec{p}_{1I}^{*NR}, \vec{p}_{2I}^{*NR}). \tag{3.16}$$

Furthermore, the arguments of the wave functions that are $\vec{p}^{*NR}_{iI/F}$, must be calculated by means of the NR limit of the Lorentz transformations of equation (3.6), that are in other words the Galilean transformations of the momenta, given in the following equation:

$$\vec{p}^{*}_{i(I/F)} = \vec{p}_i - \frac{\vec{P}_{I/F}}{A}. \tag{3.17}$$

The reader should note that one could construct the NRIA model for the vertex factors, by using the main hypotheses of the IA and the Galilean transformations of the momenta in equations (3.15) y (3.16). Finally, standard change of variables according to equations (2.4) and (2.5) show that the same results as the originally defined NRIA are obtained, as given in equations (2.8).

4 Conclusions

In this paper, we have shown that it is possible to define a relativistic generalization of the Impulse Approximation for the scattering of leptonic particles on composite (hadronic) systems. This model is deeply related to the Lorentz transformations properties of the spatial variables and of the wave functions, defining a specific form of Relativistic Quantum Mechanics (Point Form). In a subsequent work we shall derive a conserved covariant electromagnetic current, also considering the case of interacting fermions.

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